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New $\mathcal{N}$=2 Superconformal Field Theories in Four Dimensions

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Abstract: New examples of $\mathcal{N}$=2 supersymmetric conformal field theories are found as fixed points of $SU(2)$ $\mathcal{N}$=2 supersymmetric QCD. Relations among the scaling dimensions of their relevant chiral operators, global symmetries, and Higgs branches are understood in terms of the general structure of relevant deformations of non-trivial $\mathcal{N}$=2 conformal field theories. The spectrum of scaling dimensions found are all those compatible with relevant deformations of a $y^2 = x^3$ singular curve.

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1. Introduction

The study of conformal field theory (CFT) is an important starting point for the study of field theory. CFT's describe the IR behavior of asymptotically free theories and the behavior at all scales of scale invariant theories. The best known examples of non-trivial four dimensional CFT's are based on finite supersymmetric gauge theories [1-3]. These are scale invariant theories labeled by their coupling constant τ, a truly marginal operator which does not run. Their short distance behavior is not free. Other examples occur in asymptotically free theories whose IR behavior may be a non-trivial CFT or a free CFT in terms of new degrees of freedom. Examples of non-trivial CFT's from asymptotically free theories are QCD with many flavors [4], $\mathcal{N}$=1 supersymmetric QCD with many flavors [5], and the fixed points of pure $\mathcal{N}$=2 gauge theory studied in [6]. In this letter we find new examples of $\mathcal{N}$=2 CFT's as fixed points of $\mathcal{N}$=2 $SU(2)$ QCD, using the exact solution of [7], and systematize some of their properties using superconformal invariance.

In section 2 we discuss the basic conditions on non-trivial fixed points coming from conformal invariance. One consequence of these conditions is that there can be no non-trivial point with vector fields unless there are nonvanishing electric and magnetic currents in the theory. We also give a general picture of the structure of non-trivial $\mathcal{N}$=2 CFT's in terms of their relevant deformations. In section 3 we catalog the non-trivial CFT's in $\mathcal{N}$=2 $SU(2)$ QCD by determining scaling dimensions of chiral fields, their global symmetries, and Higgs branches. Four inequivalent nontrivial CFT's are found, which can be characterized as vacua where a monopole and N_f quarks simultaneously become massless, for $N_f = 1, 2, 3, 4$. The N_f=1 CFT is equivalent to the fixed point in $SU(3)$ Yang-Mills theory studied in [6]. The N_f=4 CFT is the scale-invariant $SU(2)$ 4-flavor theory studied in [7]. The scaling dimensions of relevant chiral operators of all four theories are explained in terms of the relevant deformations of a singular curve y^2=x^3.

2. Conditions from Conformal Invariance

We start by discussing four-dimensional CFT's with no supersymmetry. Fields are in representations of the conformal algebra, labeled by their scaling dimension D and their $SU(2)\times SU(2)$ Lorentz spins $(j, \tilde{\jmath})$. There is a one to one map between local operators and the states they create by acting on the conformally invariant vacuum. Primary states are those annihilated by the special conformal generators; descendant fields are created by acting with momentum generators on primary states. From the representation theory of the conformal algebra [8], unitary "chiral" primary fields, those with either j or $\tilde{\jmath} = 0$,

satisfy the inequality $D \geq j+\tilde{\jmath}+1$, with equality only for free fields. Non-chiral primary fields satisfy $D \geq j+\tilde{\jmath}+2$.

For example, consider a field strength operator $F_{\mu\nu}$ of conformal dimension D, which is the sum of two conformal primary fields $F^{\pm} = F \pm {}^*F$ with spins $(1,0)$, $(0,1)$. In the conformal algebra the states associated with the conserved currents $J^{\pm}_{\mu} = \partial^{\nu} F^{\pm}_{\mu\nu} = {}^*dF^{\pm}$ satisfy $||J^{\pm}_{\mu}\rangle|^2 = 2(D-2)$. We see that unitarity and the conformal algebra imply $D \geq 2$. Equality holds if and only if $J^{\pm} = 0$, implying the Bianchi identity and free equations of motion $dF^+ \pm dF^- = 0$ (free Maxwell theory).

If F is not free, its dimension is larger than 2 and both J^+ and J^- are not zero. Since they are descendants of different primary fields (F^+ and F^-), they are linearly independent. Therefore, both the electric current $J_e \equiv J^+ + J^-$ and the magnetic current $J_m \equiv J^+ - J^-$ are non-zero as quantum fields. We conclude that in a conformal field theory any interacting field strength must couple both to electrons and to monopoles. In particular, QED without elementary monopoles cannot have a non-trivial fixed point.[1]

Note that all Abelian gauge charges vanish in a fixed point theory (though they may still couple to massive degrees of freedom). In the case of the interacting $U(1)$ field strength F, though we have seen that its conserved electric and magnetic currents do not vanish, there is no charge at infinity associated with them, because of the rapid decay of correlation functions of F due to its anomalous dimension. This is true even if we include massive or background sources, since the long-distance behavior of the fields is governed by the conformal field theory. If, on the other hand, F were free, then we have seen that its associated conserved currents, and thus the charges, vanish. Now, however, massive sources can have long-range fields in this case since F has its canonical dimension. (We do not reach a contradiction by taking the mass of a charged source to zero since its $U(1)$ couplings flow to zero in the IR.) Non-Abelian gauge charges need not vanish in the CFT since the above arguments only apply to gauge-invariant fields or states.

The $\mathcal{N}=1$ and $\mathcal{N}=2$ superconformal algebras contain the conformal algebra, and so the above results carry over. A new feature of the superconformal algebras is their global R-symmetry. Their representations are labeled by the R-charges as well as the Lorentz spins and scaling dimension. For chiral fields the superconformal algebra implies a relation between the scaling dimension and R-charges. Since the R-symmetry is part of the conformal algebra, non-trivial supersymmetric CFT's necessarily carry non-zero R-charges.

The $\mathcal{N}=1$ superconformal algebra includes a $U(1)_R$. In theories with an $\mathcal{N}=1$ fixed point, if we can identify the $U(1)_R$ in the UV, we can determine the dimensions of chiral

[1] For a related discussion, see [9].

fields. Examples of such fixed points in QCD were first given in [5]. Alternatively, the $U(1)_R$ in the IR could be an accidental symmetry, in which case it would be difficult to find the dimensions of chiral fields.

The $\mathcal{N}$=2 superconformal algebra includes a $U(1)_R \times SU(2)_R$. In QCD the R-charges are known in the UV. If there is no anomaly in $U(1)_R$, then the theory is finite, there is a marginal coupling τ, and the dimensions are independent of τ. Examples are the non-Abelian Coulomb points in the finite $\mathcal{N}$=2 theories [2]. On the other hand, if the classical $U(1)_R$ is anomalous, there may be an accidental $U(1)_R$ in the IR, making it hard to find dimensions of chiral fields. Examples of this sort were given in [6].

2.1. Non-trivial $\mathcal{N}$=2 CFT

In preparation for our exploration of $\mathcal{N}$=2 $SU(2)$ QCD in the following section, let us examine the $\mathcal{N}$=2 case in more detail. In addition to the spin and scaling dimension, conformal fields are labeled by their $U(1)_R$ charge R, and their $SU(2)_R$ spin I. A primary state (annihilated by the superconformal generators) is the lowest component of a supermultiplet formed by applying the eight supercharges Q_i^α, $\overline{Q}_i^{\dot\alpha}$ to it.[2] From the representation theory of the $\mathcal{N}$=2 superconformal algebra [10], we learn that chiral ($\tilde{\jmath}$=0) fields ϕ satisfy $\overline{Q}_{(i}\phi_{i_1\ldots i_{2I})}$=0 and $D = 2I+\frac{1}{2}R \geq j+2I+1$. A similar relation, given by changing the sign of the R-charge holds for antichiral (j=0) fields.

The $\mathcal{N}$=2 vector multiplet U has as its lowest component a chiral primary field u with I=0 and spin $(0,0)$. Therefore $D(u) = \frac{1}{2}R(u) \geq 1$ and the field strength $F^+_{\mu\nu}$ at its second excited level has $D(F)$≥2. When this inequality is saturated, there is an extra null state at the fourth level giving dF^+=0. In $\mathcal{N}$=2 superfield notation, a chiral superfield U,

$$\overline{D}_{\dot\alpha i}U = 0, \tag{2.1}$$

with $R(u)$=2 satisfies

$$D^{\alpha(i}D_\alpha^{j)}U = 0. \tag{2.2}$$

In this case the field is free, the null state equations for u and $\overline{u}$ being equivalent to the vacuum Maxwell equations. On the other hand, for an interacting vector multiplet ($D(u)$>1) these states are no longer null, and we again conclude that the corresponding electric and magnetic currents cannot vanish as quantum fields.

$\mathcal{N}$=2 QCD has a moduli space of inequivalent vacua, composed generically of "branches" with some numbers n_V (massless) $U(1)$ vector multiplets and n_H massless

[2] α and $\dot\alpha$ are Lorentz indices, i is the $SU(2)_R$ index.

neutral hypermultiplets. Branches with n_H=0 are called Coulomb branches, with n_V=0 Higgs branches, and the other cases mixed branches. These branches may intersect along complex submanifolds, corresponding to phase transitions. The Higgs branch is determined classically due to a nonrenormalization theorem [11], which follows from thinking of the bare masses and the strong-coupling scale Λ as the scalar components of $\mathcal{N}$=2 vector superfields.[3] In the general $\mathcal{N}$=2 effective action [12] it is found that the scalar components of vector multiplets do not appear in the hypermultiplet metric, forbidding any mass or Λ-dependent corrections to the Higgs branch. A similar argument also shows that the Coulomb branch cannot receive any squark–vev dependent corrections, and that the mixed branches have the structure of a direct product of a Higgs and Coulomb branch.

This also implies that there are no non-trivial fixed points even at strong coupling on a Higgs branch, except, perhaps, at points where it joins a Coulomb branch. Generic points on the Coulomb branch are free $\mathcal{N}$=2 $U(1)^{n_V}$ gauge theory in the IR. Along certain submanifolds of complex codimension one in moduli space the low-energy theory contains in addition a massless charged hypermultiplet, giving massless $\mathcal{N}$=2 QED as the IR theory. In terms of the original gauge fields the light matter fields may be magnetic monopoles or dyons of various charges. Where these submanifolds intersect there will be two or more massless charged hypermultiplets. When the various massless hypermultiplets at some point are all mutually local the low-energy theory is simply $\mathcal{N}$=2 electrodynamics with massless matter, written in terms of the gauge fields to which the matter fields couple locally. The term "mutually local" as used here means simply that there is an electric-magnetic duality transformation in the low-energy $U(1)$ to a description of the physics in which no fields carry magnetic charge. The new phenomenon studied in [6] occurs when the massless states at some point are mutually nonlocal, which, by the preceding discussion, will be a nontrivial $\mathcal{N}$=2 superconformal field theory.

We can deform any fixed point on the Coulomb branch by vevs of $\mathcal{N}$=2 vector superfields to another point on the Coulomb branch, or by relevant operators. For simplicity, consider the case n_V=1, so that at a generic point on the Coulomb branch we have a free Maxwell theory described by a free $U(1)$ $\mathcal{N}$=2 vector superfield U, with $D(u)$=1. The effect of adding a relevant operator appears in the low energy theory as a variation of the prepotential. An example is a shift of the mass term for the underlying quarks in the microscopic QCD Lagrangian. The leading order operator at a given point on the moduli space can be found by expanding the resulting variation of the prepotential in U. The constant term obviously does not contribute. The linear term, $\int d^4\theta\, U$, is a total derivative.[4] This follows by using the chirality (2.1) and null vector (2.2) conditions. Therefore,

[3] This is possible since one knows how to write explicit Lagrangian with weakly gauged $U(1)$'s whose scalar part appears as the bare mass or Λ.

[4] $\int d^4\theta$ is an integral over half of $\mathcal{N}$=2 superspace.

the leading effect of the mass term is to change the effective coupling τ (the coefficient of U^2). In $\mathcal{N}$=2 QED (*i.e.* along the codimension one submanifolds of the Coulomb branch where a charged hypermultiplet becomes massless) a mass term can again be absorbed in a shift of U, and again the term linear in U does not have an effect.

At a non-trivial fixed point, on the other hand, U is chiral, but no longer satisfies (2.2). Expanding around U=0, the leading effect of a mass term is then

$$\delta\mathcal{L} = \int d^4\theta\, mU, \tag{2.3}$$

implying $D(m) + D(u) = 2$.

Alternatively, we can think of all the parameters as background gauge fields or as weakly coupled propagating gauge fields. This turns the relevant operator deformations into deformations along the Coulomb branch of these new gauge fields. The masses can be thought of as the background values of the scalar components of $\mathcal{N}$=2 vector multiplets coupling to the conserved flavor currents. Since conserved charges are dimensionless, their conserved currents have dimension three. These, in turn, couple to the vector field in the multiplet, whose dimension is thus required to be one. The $\mathcal{N}$=2 algebra relates this to the dimension of the scalar component, leading to the requirement that the masses have dimension one.

This conclusion depended on the flavor symmetry being a global symmetry, so we could think of it as being arbitrarily weakly gauged. The conclusion is quite different if the global symmetry vanishes in the conformal field theory. This can happen in practice when a global $U(1)$ symmetry looks at low energies like a global $U(1)$ gauge transformation (like lepton number in QED, which coincides with the global electric charge). Then, at a non-trivial fixed point of the effective $U(1)$ theory, the gauge charge and thus the global symmetry will vanish, as we have seen above. There may also be other mechanisms by which a global symmetry of a microscopic Lagrangian can become trivial at a conformal fixed point.

When the conserved currents do not act in the CFT (*i.e.*, their charges vanish for all fields in the CFT) the superfield weakly gauging the symmetry can couple to the CFT only through irrelevant operators. For example, consider weakly gauging a $U(1)$ symmetry with gauge field V whose dimension is one. Its leading coupling to the CFT is through the irrelevant operator

$$\Lambda^{-\delta} \int d^4\theta\, VU, \tag{2.4}$$

where $D(u) = 1{+}\delta$. Then, as we give an expectation value $\langle v \rangle$ to the scalar component of V, the CFT is deformed by the relevant or marginal operator

$$\delta\mathcal{L} = \frac{\langle v \rangle}{\Lambda^\delta} \int d^4\theta\, U, \tag{2.5}$$

if $\delta{\leq}1$. Therefore, we identify the singlet mass parameter $m = \langle v \rangle \Lambda^{-\delta}$, which has dimension $1{-}\delta$. If $D(u){>}2$ ($\delta{>}1$) the deformation (2.5) is irrelevant.

This can be summarized as follows: deforming the CFT by a gauge superfield V with dimension one leads to explicit breaking of scale invariance, while deforming by a field U with dimension larger than one leads to spontaneously broken scale invariance. The deformation (2.5) could arise with V one of the elementary gauge fields in a non-Abelian theory, as in the $SU(3)$ Yang-Mills example of [6], or from singlet mass terms for elementary hypermultiplets, as in the $SU(2)$ QCD examples to be discussed in section 3.

This picture can be generalized to include many U's, in which case the CFT has a set of relevant or marginal parameters (u_i, m_i) satisfying $1 < D(u) \leq 2$ and $D(u_i) + D(m_i) = 2$, as well as parameters m_A governing the coupling to any non-Abelian global symmetries. Examples of this sort occur in $n{\geq}4$ $SU(n)$ Yang-Mills [6].

3. Examples in SU(2)

In this section we study the non-trivial fixed points occurring in $\mathcal{N}{=}2$ QCD with $SU(2)$ gauge group. To see that such vacua might be expected in this theory, note that they can arise at most in complex codimension two in moduli space, explaining their absence in the $SU(2)$ Yang-Mills solution of [13], where the moduli space is the complex $\tilde{u}$-plane. When matter multiplets in the fundamental are included, however, the bare masses appear as parameters in the theory. By tuning these parameters we can find values for which various subsets of the singular points in the $\tilde{u}$-plane coincide. If the colliding singularities correspond to mutually nonlocal states, there will be a new CFT. We present a complete catalog of such points arising in $\mathcal{N}{=}2$ QCD with $SU(2)$ gauge group, some of which are manifestly inequivalent to the CFT discovered in [6].

At a point in parameter space where some singular points in the moduli space coincide, there will be new, interacting physics if the massless states associated to the two colliding singularities are mutually nonlocal. This can be determined by a local monodromy computation, but we will propose a much simpler criterion. As one varies the parameters, if at some point in moduli space an enlarged set of mutually local states become massless then the dimension of the Higgs branch should increase for this special value of the parameters

(as happens, for example, when the bare masses are tuned to coincide for two or more quarks). Thus, if two singularities collide for some value of the parameters and the Higgs branch does not change at this value, the colliding points correspond to mutually nonlocal states. This Higgs branch criterion is effective because the structure of the Higgs branch is determined classically due to the nonrenormalization theorem discussed in the last section.

The Higgs branch criterion immediately suggests the following structure of fixed points. Choose the bare masses for all N_f quarks to be the same, M, giving an unbroken $U(N_f)$ global flavor symmetry, and an $N_f{-}1$ dimensional Higgs branch.[5] There are then three singular points in the (finite) $\tilde{u}$-plane with the massless hypermultiplets at one transforming in the fundamental representation of the flavor symmetry and those at the other two invariant. For large M these are interpreted as (one component of) the original quarks at $\tilde{u}{\sim}M^2$, and the monopole and dyon states of the $SU(2)$ Yang-Mills theory obtained after integrating out the massive quarks. By tuning M so that a "quark" point coincides with one of the other two, we can find new singularities. The discussion of the previous paragraph tells us that if the coincidence occurs for nonzero M, so that the global symmetry and Higgs branch are not modified from their form at generic M, the low-energy theory should be an interacting CFT. We will call such points $(N_f,1)$ points, indicating that N_f mutually local states are massless together with one state nonlocal with respect to them.

We thus predict the existence of these new CFT's. In the rest of this section we show that these points exist for $N_f{\leq}4$, and that this list is comprehensive. The exact solution is written in terms of a cubic plane curve, describing a torus as a branched cover of the x-sphere. The coefficients of the cubic polynomial are themselves polynomials in $\tilde{u}$ and the masses M_i, $i{=}1,\ldots,N_f$. Singular points in the $\tilde{u}$-plane correspond to degenerations of the torus, or equivalently to the coincidence of roots of the polynomial. The interacting points in codimension two will arise when all three of the roots at finite x coincide. Because three branch points coincide at these points, two intersecting cycles on the torus are going to zero, and thus the light hypermultiplets near these points are mutually nonlocal. In the vicinity of such a point, we can write the curve in terms of local coordinates u, m, $\widetilde{x}$ as $y^2 = \widetilde{x}^3{-}f(u,m)\widetilde{x}{-}g(u,m)$, where f,g are polynomials such that $f(0,0) = g(0,0) = 0$. Keeping only the lowest-order terms in f, g gives a quasihomogeneous polynomial, leading to an assignment of R-charges and hence to a prediction of the anomalous dimensions of some of the chiral operators in the CFT.

[5] For $M{=}0$ the flavor symmetry is enhanced to $SO(2N_f)$ and the Higgs branch is correspondingly enlarged.

3.1. $N_f = 1$

The one-flavor curve[6] $y^2 = x^2(x-\tilde{u}) + 2Mx - 1$ is singular at the zeros of the discriminant $\Delta_1 = -4\tilde{u}^3+4\tilde{u}^2M^2+36\tilde{u}M-32M^3-27$. Special points can arise when two or more of the zeros of Δ_1 coincide. This must necessarily happen, since the discriminant of Δ_1 considered as a cubic in $\tilde{u}$ is holomorphic in M and hence vanishes somewhere. Indeed the special point occurs at $M{=}\frac{3}{2}\omega$, $\tilde{u}{=}3\omega^{-1}$, for which three branch points coincide at $x{=}\omega^{-1}$, where $\omega^3{=}1$.[7] Expanding about this point (taking $\omega{=}1$) and rewriting the curve in terms of the shifted variables $M = \frac{3}{2}+m$, $\tilde{u} = 3+2m+u$, and $x = \frac{1}{3}\tilde{u}+\widetilde{x}$, we have

$$y^2 = \widetilde{x}^3 - 2(m+u)\widetilde{x} - (u + \tfrac{4}{3}m^2), \tag{3.1}$$

where we have dropped higher-order terms in m and u which are necessarily smaller than those shown as $m, u{\to}0$. Furthermore, in this limit we must assign relative scaling dimensions $D(x) : D(m) : D(u) = 1 : 2 : 3$ in order to see the cubic singularity. Then the ux and m^2 terms are negligible near the CFT point, and we find the curve

$$y^2 = \widetilde{x}^3 - 2m\widetilde{x} - u. \tag{3.2}$$

We discuss how the apparent scaling dimensions of the coefficients of the curve translate into scaling dimensions of operators in the CFT at the end of this section.

The curve (3.2) is identical to the expansion of the $SU(3)$ Yang-Mills curve around the point discovered in [6]. Also, neither theory has a Higgs branch or a flavor symmetry. One difference is that here m appears as a parameter whereas in [6] its place was taken by the background value of a field; as discussed above, this difference is not essential. Indeed, it was shown in [14] that these two CFT's are the same by realizing them as different limits related by S-duality of the $\mathcal{N}{=}2$ $SU(3)$ theory with one adjoint flavor. We refer to this class of special point as $(1,1)$ indicating that it arises when the singularities corresponding to two, mutually nonlocal massless states coincide.

[6] We set $\Lambda^{4-N_f}{=}8$ by a choice of units. The appropriate powers of Λ can easily be reinstated using the R-symmetry charges.

[7] The bare mass breaks a $\mathbb{Z}_3$ symmetry acting on the $\tilde{u}$-plane giving three singular values of m with identical structure. In terms of the identification of the massless states at large M, these three points correspond to the quark, monopole, and dyon becoming massless in pairs.

3.2. $N_f = 2$

For two flavors the curve is $y^2 = (x^2-1)(x-\tilde{u}) + 2M_1M_2x - (M_1^2+M_2^2)$. The four values of $\tilde{u}$ at which this is singular are given by the vanishing of its discriminant Δ_2. To determine when these zeros collide we can study the discriminant in $\tilde{u}$ of Δ_2, which has a factor corresponding to the lines $M_1 = \pm M_2$ of coincidence of the two electron points when their bare masses are degenerate, and another factor Δ along which there are additional coincidences. We can obtain Δ more directly by requiring that the two-flavor curve be an exact cube $y^2 = (x-\frac{1}{3}\tilde{u})^3$, yielding $\tilde{u}^2 = 6M_1M_2-3$ and $\tilde{u}^3+27\tilde{u} = M_1^2+M_2^2$. Eliminating $\tilde{u}$ gives a locus $\Delta(M_i) = 0$ of singularities in M-space.

Taking the scaling limit at large M_2 (and vanishing Λ) leads to the $(1,1)$ point found above. By continuity, we expect a line of such points along any branch of Δ, extending from the large-M region until a new singularity appears. There are two possibilities here: either the low-energy theory remains constant as we move M along this curve and tune $\tilde{u}$ to the singular value, or they are connected by an exactly marginal operator. The former seems more likely, since there is no continuous parameter in the $(1,1)$ curve (3.2) which could correspond to the parameter of the marginal deformation. In particular, the coefficients in (3.2) can be absorbed in simple redefinitions (rescalings) of m and u.

Additional singularities occur when Δ intersects one of the lines $M_1 = \pm M_2$, or when it is itself singular. There are singularities of Δ at the four solutions of $M_1^2+M_2^2 = 0$, $M_1M_2 = -4$. These points contain no new physics; the apparent singularity is due to the existence of two different values of $\tilde{u}$ for the same M_i at which the theory is singular. The other singularities of Δ in M-space occur only at $M_1 = \pm M_2$, for the values $M_1^2 = \pm 2$. These points *do* correspond to new physics since a $U(2)$ flavor symmetry is unbroken by the bare masses, and the Higgs branch of the theory (which is constant along the line $M_1 = \pm M_2$) does not change at this point. This indicates a nonlocal collision, and also shows that this CFT cannot be equivalent to the $(1,1)$ theory.

To study this point, rewrite the two-flavor curve in terms of the flavor invariants $M \equiv \frac{1}{2}\sum_i M_i$ and $C_2 \equiv \sum_i (M_i - M)^2$, in terms of which the point in question is $M{=}\sqrt{2}$, $C_2{=}0$. Expand about this point, defining the shifted variables $M = \sqrt{2}+m$, $\tilde{u} = 3+2\sqrt{2}m-\frac{1}{3}m^2+u$, and $x = \frac{1}{3}u+\widetilde{x}$, to give the curve

$$y^2 = \widetilde{x}^3 - 2u\widetilde{x} - \frac{4\sqrt{2}}{3}mu + \frac{16\sqrt{2}}{27}m^3 - 2C_2. \tag{3.3}$$

The structure of the theory around this point can be read from (3.3). The dimensions of the relevant couplings are in the ratios $D(m) : D(u) : D(C_2) = 1 : 2 : 3$. In the vicinity of the special point there are distinguished deformations. Varying the singlet mass m with

C_2=0 preserves the $U(2)$ global symmetry; the singularity in the u-plane splits, one point corresponding to two massless states in the **2** and the other to a singlet. The Higgs branch is unchanged under this deformation. Further, one expects to find $(1,1)$ points in the vicinity of the $(2,1)$ point. Indeed, (3.3) factors as a cubic for $4m^3{=}9\sqrt{2}C_2$, $u{=}-\frac{1}{3}m^2$; this is the local form of Δ about this point.

As an illustration of the Higgs branch criterion in action, consider the case when the two quarks have a common mass M. For large M there will be a 2-quark point at $\tilde{u}{\sim}M^2$, and a monopole and a dyon point at $\tilde{u} \sim \pm 1$. There is a Higgs branch isomorphic to $\mathbb{C}^2/\mathbb{Z}_2$ attached to the 2-quark point, and the low-energy theory has an unbroken global $U(2)$ flavor symmetry; the monopole and dyon points have no Higgs branches and only a $U(1)$ baryon number. Classically, decreasing M gives no qualitatively new physics until M=0, at which point the global symmetry is enhanced to $SO(4)$ and two Higgs branches appear, both isomorphic to $\mathbb{C}^2/\mathbb{Z}_2$. This must also be true quantum-mechanically by the nonrenormalization theorem. The Higgs branch criterion implies that collisions of monopole points at M=0 where the dimension of the Higgs branch changes will correspond to a vacuum with only mutually local states.

On the other hand, from the exact solution, we expect three singularities as we decrease M: two where the 2-quark point coincides with a monopole or dyon point, and one where the monopole and dyon points meet. Naively, all these singularities involve massless mutually non-local hypermultiplets. As shown above, the first two of these meetings take place at $M = \pm\sqrt{2}$ and $\tilde{u}$=3, and indeed involve mutually nonlocal states since three branch points coincide at that point. The other singularity occurs at M=0, where the curve becomes $y^2 = (x^2{-}1)(x{-}\tilde{u})$. Here there are two singularities in the $\tilde{u}$-plane, corresponding to massless monopole states in one spinor of $SO(4)$ at $\tilde{u}$=1, and massless dyons in the other spinor of $SO(4)$ at $\tilde{u}{=}-1$ [7]. We see directly from the curve that the vacua involve only mutually local massless particles. The naive picture that this vacuum arose from the collision of a monopole and dyon point, and therefore should have involved mutually nonlocal massless states, is incorrect because it did not take into account the possibility that as $M{\to}0$ the monodromies around the monopole or dyon point are conjugated by the 2-quark point monodromy as that point moves on the $\tilde{u}$-plane. Indeed, one can check that this is what happens: the 2-quark singularity passes between the monopole and dyon points as $M{\to}0$.

3.3. $N_f = 3$

The curve here is $y^2 = x^2(x{-}\tilde{u}) - (x{-}\tilde{u})^2 - A(x{-}\tilde{u}) + 2Bx - C$, where $A = M_1^2{+}M_2^2{+}M_3^2$, $B = M_1M_2M_3$, and $C = M_1^2M_2^2{+}M_2^2M_3^2{+}M_1^2M_3^2$. To find $(1,1)$ points

set $y^2 = (x-\frac{1}{3}\tilde{u}-\frac{1}{3})^3$ and eliminate $\tilde{u}$ to find a surface $\Delta(M_i) = 0$. Along this surface in M-space, the generic singular form of the three-flavor curve describes $(1,1)$ vacua. The surface is singular along a submanifold where two of the bare masses coincide; for these masses there are $(2,1)$ points with the appropriate Higgs branch and $U(2)$ global symmetry. This submanifold is itself singular when all of the masses are equal, leading to a $(3,1)$ point occurring at $M_1{=}M_2{=}M_3{=}1$ with a $U(3)$ flavor symmetry. A detailed study of Δ shows that there are no other singularities, thus we expect no other nontrivial CFT's to arise.

The nonzero bare masses at the $(3,1)$ point mean the Higgs branch is not changed and hence there is a new non-trivial CFT here. To study this point rewrite the three-flavor curve in terms of the invariants $M \equiv \frac{1}{3}\sum_i M_i$, $C_2 \equiv \sum_i (M_i - M)^2$, and $C_3 \equiv \sum_i (M_i - M)^3$, in terms of which the point in question is $M{=}1$, $C_2{=}C_3{=}0$. We expand about this point and shift $M = 1{+}m$, $\tilde{u} = 2{+}3m{+}m^2{-}\frac{5}{6}m^3{+}u$, and $x = \frac{1}{3}{+}\frac{1}{3}u{+}\widetilde{x}$. Writing the curve in terms of the shifted variables and dropping higher orders we have

$$y^2 = \widetilde{x}^3 - 2(mu + C_2)\widetilde{x} - u^2 - \frac{m^3 u}{3} + \frac{m^6}{108} - \frac{2m^2 C_2}{3} + \frac{8C_3}{3}. \tag{3.4}$$

The relevant couplings now satisfy $D(m) : D(u) : D(C_2) : D(C_3) = 1 : 3 : 4 : 6$. There is a surface of $(1,1)$ points coinciding with the local form of Δ.

3.4. $N_f \geq 4$

For $N_f{=}4$, the bare masses M_i break the exact scale symmetry of the massless theory [7]. This means that an overall scaling of the masses is not a true parameter of the theory, so that one cannot expect a $(4,1)$ point to appear at codimension four. On the other hand, scale invariance means the classical coupling τ is a parameter in the theory; the coefficients of the polynomial are modular functions of τ. One expects to find (though we did not check this) a hypersurface in M-space, varying with τ, along which there are $(1,1)$ points, singularities on this corresponding to $(2,1)$ and then $(3,1)$ points. We did look for the possible occurrence of a $(4,1)$ point at a particular value of τ, and found that this does not happen. Of course, we already know of a new conformal fixed point that occurs for $N_f{=}4$, namely the non-Abelian Coulomb point at $M_i{=}\tilde{u}{=}0$. By a series of shifts and rescalings, the curve near this point can be put into the form

$$y^2 = \widetilde{x}^3 + (u^2 + uC_2 + C_2^2 + C_4 + C_4')\widetilde{x} + (u^3 + u^2 C_2 + \ldots + C_6), \tag{3.5}$$

where we have suppressed the coefficients, which are various modular functions of the coupling τ, and where the C_i are the mass-invariants of the $SO(8)$ flavor group: $C_2 \equiv$

$\sum_i M_i^2$, $C_4 \equiv \sum_{i<j} M_i^2 M_j^2$, $C_4' \equiv \prod_i M_i$, and $C_6 \equiv \sum_{i<j<k} M_i^2 M_j^2 M_k^2$. In this case the R-charges of the relevant and marginal couplings u, τ, C_2, C_4, C_4', and C_6, are just proportional to their classical R-charges.

For $N_f{>}4$ the $SU(2)$ theory is no longer asymptotically free, and instead has a free non-Abelian Coulomb phase at $\tilde{u}{=}0$. The Coulomb branch can be described in the vicinity of this point (by breaking to $SU(2)$ from an asymptotically-free $SU(N_c)$ gauge theory with N_f flavors, as described in [15]) by the curve

$$y^2 = (x^2 - \tilde{u})^2 - \Lambda^{4-N_f} \prod_{i=1}^{N_f} (x - M_i). \tag{3.6}$$

This description is only reliable for x, $\tilde{u}$, and $M_i \ll \Lambda$; values of the parameters and moduli outside this region probe physics that depends on the UV regulator. For small but non-zero $M_i{\sim}M$, the monopole and dyon singularities of the effective $SU(2)$ Yang-Mills theory at energies less than M occur at $\tilde{u}^2 \sim \Lambda^4 (M/\Lambda)^{N_f}$. Thus the quark singularities at $\tilde{u}{\sim}M^2$ can only approach the monopole and dyon singularities for $M{\sim}\Lambda$ where the curve (3.6) is no longer valid. We conclude that there are no new interacting vacua in $\mathcal{N}{=}2$ $SU(2)$ QCD with $N_f{>}4$.

3.5. The R-Symmetry

The quasihomogeneous polynomials (3.2-5) determine the R-charges of the various couplings, and hence their dimensions, up to an overall scaling. A constraint which fixes the overall normalization of the R-charges follows from the requirement that the Kähler potential $K = \mathrm{Im}(A_D \overline{A})$ have dimension two. This, in turn, means that a has dimension one. Using the representation of a as a contour integral on the torus specified by the cubic curve, we have $a \sim (u/y) d\widetilde{x}$, which then leads to a normalization of the dimensions.

N_f	y	$\widetilde{x}$	u	m (τ)	m_A	a
1	3/5	2/5	6/5	4/5		1
2	1	2/3	4/3	2/3	1	1
3	3/2	1	3/2	1/2	1	1
4	3	2	2	0	1	1

Table 1: Scaling weights of the couplings at $(N_f, 1)$ points for various N_f. The adjoint masses are defined by $m_A = (C_j)^{1/j}$. For $N_f{=}4$, τ plays the role of the scalar mass m.

We collect the data on the four CFT's, normalized in this way, in Table 1. The non-integral spectra of R-charges demonstrate that the R-symmetry in the $\mathcal{N}$=2 superconformal algebra is not a symmetry of the classical Lagrangian for $N_f \neq 4$. In addition, the spectrum of dimensions shows that these are not free field theories. The first example of an $\mathcal{N}$=2 SCFT in four dimensions exhibiting this property is the $(1,1)$ theory found in $SU(3)$ Yang-Mills theory [6]. A number of properties are immediately clear from the table:

1. The dimension of $m_A = (C_j)^{1/j}$ is one (in the cases in which it is defined).
2. The dimension of u satisfies $1 < D(u) \leq 2$.
3. The dimensions of u and m satisfy $D(u)+D(m) = 2$.

The explanation of properties 1–3 in terms of $\mathcal{N}$=2 CFT was given in section 2. Property 1 follows from the fact that the adjoint mass couples to conserved flavor currents which act in the CFT. Property 2 reflects the fact that these are interacting CFT's with u (the vev of) a relevant operator. Property 3 follows from the form of the relevant coupling (2.3).

We can give no explanation of the spectrum of dimensions in Table 1 from first principles; however, assuming a cubic form for the singularity, one can recover the information in Table 1. In particular, assume that the curve around the CFT point is given by $y^2 = x^3 - fx - g$ where f, g are polynomials in u, m, and m_A, which correspond to relevant operators. Normalizing the dimensions by demanding that $a \sim (u/y)dx$ have dimension one, and using properties 1–3 above, gives $D(m_A)$=1, $D(m)$=2$-D(u)$, $D(f)$=4$D(u)-4$, $D(g)$=6$D(u)-6$, and $D(x)$=2$D(u)-2$. Either f or g must include a term u^α for α=1, 2, . . ., since otherwise the curve would be singular for all u when m=0, implying that u is not, in fact, a relevant operator as we had assumed. Assuming first $g \sim u^\alpha$ one finds that only the values α=1, 2, 3 are compatible with with the above constraints. For α=1 one finds the $(1,1)$ curve; for α=2 the $(3,1)$ curve; and for α=3 the $(4,1)$ curve. On the other hand, assuming $f \sim u^\beta$ gives the $(2,1)$ curve for β=1; coincides with the α=3 case when β=2; and no other β are allowed. We thus recover precisely the singular curves found above. Furthermore, many (though not all) of the coefficients of the various terms in the curves can be taken to 1 by appropriate rescalings and shifts of u, m, and m_A.

One could imagine developing in this way an algebraic classification of four-dimensional $\mathcal{N}$=2 CFT's.

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